OML2 - Chapitre 6: Compléments sur les intégrales

Gabriel Soranzo

Nous avons vu

$$g(u)' = u'g'(u) \text{ c.a.d. } g(ut(t))' = u'(t)g'(u(t))$$

donc

$$\underbrace{g}_{F}(u) = \int u' \underbrace{g'}_{f}(u) \text{ c.a.d. } g(u(t)) = \int u'(t)g'(u(t))\,dt$$

donc on posant $F = g$ et donc $f = g'$ alors

$$F(u) = \int u'f(u) \text{ c.a.d. } F(u(t)) = \int u'(t)f(u(t))\,dt$$

donc

$$\begin{aligned}\int_a^b u'(t)f(u(t))\,dt &= [F(u(t))]_a^b \\ &= F(u(b)) - F(u(a)) \\ &= [F]_{u(a)}^{u(b)} \\ &= \int_{u(a)}^{u(b)} f(t)\,dt\end{aligned}$$

On a donc au final

$$\boxed{\int_a^b f(u(t))u'(t)\,dt = \int_{u(a)}^{u(b)} f(x)\,dx}$$

En termes moins rigoureux

$$\boxed{\int_a^b f(u)u'\,dt = \int_{u(a)}^{u(b)} f(u)\,du}$$

Exemple 0.1
Calculons:

$$\begin{aligned}
&\int_0^{\pi/3} \cos^2(t)\sin(t)\,dt \quad \text{On pose } \left\{\begin{array}{l} u = \cos(t) \\ u' = -\sin(t) \end{array}\right. \\
&= -\int_0^{\pi/3} (\cos(t))^2(-\sin\,t)\,dt = -\int_0^{\pi/3} (u(t))^2 u'(t)\,dt \\
&= -\int_0^{\pi/3} f(u(t))u'(t)\,dt \quad \text{Avec } f = x^2 \\
&= -\int_{u(0)}^{u(\pi/3)} f(x)\,dx \quad \text{Avec la formule de changement de variables} \\
&= -\int_1^{1/2} x^2\,dx \quad \text{car } \left\{\begin{array}{l} u(0) = \cos(0) = 1 \\ u(\pi/3) = \cos(\pi/3) = 1/2 \end{array}\right. \\
&= -\left[\frac{x^3}{3}\right]_1^{1/2} = -\left(\frac{1/8}{3} - \frac{1}{3}\right) = -\left(-\frac{7}{24}\right) = \frac{7}{24}
\end{aligned}$$

Exemple 0.2
Calculons:

$$\begin{aligned}
&\int_0^4 \frac{dt}{1+\sqrt{t}} \quad \text{on pose } \left\{\begin{array}{l} u = \sqrt{t} \\ u' = \frac{1}{2\sqrt{t}} \end{array}\right. \\
&= \int_0^4 \frac{2\sqrt{t}}{1+\sqrt{t}} \frac{1}{2\sqrt{t}}\,dt = \int_0^4 \frac{2u}{1+u} u'\,dt \\
&= \int_{u(0)}^{u(4)} \frac{2x}{1+x}\,dx = 2\int_0^2 \frac{x}{1+x}\,dx \quad \left\{\begin{array}{l} u(0) = \sqrt{0} = 0 \\ u(4) = \sqrt{4} = 2 \end{array}\right. \\
&= 2\int_0^2 1 - \frac{1}{1+x}\,dx \quad \text{(DSE)} \\
&= 2\left[x - \ln(1+x)\right]_0^2 = 2((2-\ln 3) - (0 - \ln 1)) = 2(2-\ln 3)
\end{aligned}$$